Homework 1

1. **Tightly Estimating Summations.** Use integration to tightly estimate the following expressions.

   (a) **(15 points)** $S_n = \sum_{i=1}^{n} \frac{1}{i}$,
   Solution.

(b) **(15 points)** $S_n = \sum_{i=1}^{n} \ln i$,
Solution.

(c) **(15 points)** $S_n = n!$
(Remark: Recall that $n! = \prod_{i=1}^{n} i$)
Solution.

2. **Trapezoid Rule.** In the lecture, we saw that if f is a concave upwards function then the following is true.

$$\frac{f(x-1)+f(x)}{2} \geqslant \int_{x-1}^{x} f(t)\,\mathrm{d}t$$

   (a) **(20 points)** Prove that, for a concave upwards function f, we have

$$f(1)+f(2)+\cdots+f(n) \geqslant \frac{f(1)+f(n)}{2} + \int_{1}^{n} f(t)\,\mathrm{d}t$$

   Solution.

(b) **(10 points)** Use this result to lower-bound the sum

$$S_n = \sum_{i=0}^{n-1} a^i,$$

where a is a positive real number.
Solution.

3. **Understanding Joint Distribution.** Recall that in the lectures we considered the joint distribution $(\mathbb{T}, \mathbb{B})$, where $\mathbb{T}$ represents the time I wake up in the morning, and $\mathbb{B}$ represents whether I have breakfast or not. The following table summarizes the joint probability distribution.

t	b	$\mathbb{P}[\mathbb{T} = t, \mathbb{B} = b]$
4	T	0.03
4	F	0
5	T	0.02
5	F	0
6	T	0.30
6	F	0.05
7	T	0.20
7	F	0.10
8	T	0.10
8	F	0.08
9	T	0.05
9	F	0.05
10	T	0
10	F	0.02

Calculate the following probabilities.

(a) **(10 points)** $\mathbb{P}[\mathbb{T} \leqslant 7, \mathbb{B} = T]$,
Solution.

(b) **(10 points)** $\mathbb{P}\left[\mathbb{T} \leqslant 7\right]$, and
Solution.

(c) **(10 points)** $\mathbb{P}\left[\mathbb{B} = \mathsf{T} \mid \mathbb{T} \leqslant 7\right]$.
Solution.